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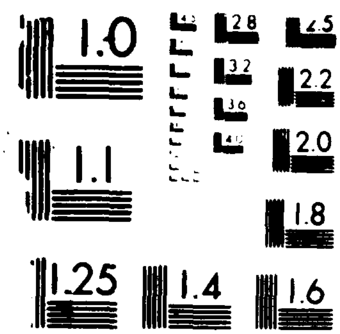

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BOUNDARY STABILIZATION OF THIN ELASTIC PLATES[*†]

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1. **Introduction.** In this paper we shall consider the question of *uniform stabilization* of thin, elastic plates through the action of forces and moments on the edge of the plate (or on a part of the edge of the plate). Two particular plate models will be considered: The classical fourth order *Kirchhoff model*, but incorporating rotational inertia, and the "sixth order" *Mindlin-Timoshenko model*. The difference in the two models, from a physical point of view, is that the M-T model incorporates *transverse shear* effects while the Kirchhoff model does not. Actually, the M-T model is a hyperbolic system of three coupled second order partial differential equations in two independent variables. The unknowns, denoted by w, ψ, φ are the vertical component w of displacement and angles ψ, φ which are measures of the amount of transverse shear. The three equations are coupled through terms which are multiples of a factor K called the *coefficient of elasticity in shear*. It is well known that, as $K \to \infty$, the limit of $w = w_K$ is *formally* a solution of the fourth order Kirchhoff plate equation (2.1) below (see Remark 2.1 below. The manner in which certain *conservative* M-T systems converge to the corresponding Kirchhoff systems has been made precise in J. LAGNESE-J.L. LIONS [10, Chapter II].)

For the M-T model, it will be shown that one can find simple, natural

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feedback laws for the edge forces and moments which lead to an *explicit*, *uniform* decay rate $\omega=\omega(K)$ for solutions to the closed-loop system (under some geometric conditions on the shape of the plate). It will be seen, however, that $\omega(K)\to 0$ as $K\to\infty$. On the other hand, we can show that, as $K\to\infty$, the limit of $w=w_K$ exists and is *the* solution to a *dissipative* closed-loop Kirchhoff system. One may then prove *directly* that the limit system likewise has a uniform decay rate ω^*. Thus we have the puzzling situation that the limit of the decay rate $\omega(K)$ is *not* the decay rate ω^* for the limit system.

A related paradox was encountered in [10] in connection with *exact boundary controllability* of the M-T model. There it was shown that solutions of the M-T model are exactly controllable in a finite time $T_0=T_0(K)$, where $T_0(K)\to\infty$ as $K\to\infty$. Nevertheless, the limit of the M-T system is a Kirchhoff system which can be shown to be exactly controllable in a finite time T_0^*. It should be remarked, however, that these anomolies *may* be a result of the *methods* employed rather that anything intrinsic to the models themselves.

For related work on uniform boundary stabilization of (mainly) hyperbolic equations we refer to G. CHEN [1,2], V. KOMORNIK-E. ZUAZUA [6], J. LAGNESE [7,8,9], I. LASIECKA-R TRIGGIANI [11], and J.L. LIONS [12]. For results on stabilization of *beam* problems with end damping, see G. CHEN *et al* [3,4] and J.U. KIM and Y. RENARDY [5].

2. Models to be Considered. Consider a homogeneous, isotropic, thin plate of uniform thickness h. Points within the body will be represented by rectangular coordinates (x,y,w). It is assumed that the plate has a *middle surface* midway between its faces which, in equilibrium, occupies the region

Ω of the plane $w=0$. We denote by $w(x,y,t)$ the w-component of the displacement vector at time t of the point which, when the plate is in equilibrium, has coordinates $(x,y,0)$.

It is customary in thin plate theory to assume that the transverse normal stress σ_w is negligable compared to the other stresses. It is further assumed that all components of displacement are "sufficiently small" to justify linearization. (The reader may consult any number of works for details of the linear theory of elastic plates. A particularly accessible source is the book of K WASHIZU [13]. Heuristic derivations of a number of plate models, both linear and nonlinear, may be found in Chapter I of [10].)

The *Kirchhoff model* results if one further assumes that the linear filiments of the plate initially perpendicualr to the middle surface remain straight and perpendicular to the deformed middle surface and undergo neither contraction nor extension. (Thus *transverse shear* is neglected.) These assumptions lead to the following partial differential equation for $w(x,y,t)$:

$$\rho h w'' - \frac{\rho h^3}{12} \Delta w'' + D\Delta^2 w = 0 \quad \text{in } \Omega\times(0,\infty), \tag{2.1}$$

where $' = \frac{\partial}{\partial t}$, Δ is the ordinary Laplacian in (x,y) variables, ρ is the surface density per unit area and $D=Eh^3/12(1-\mu^2)$ is called the *modulus of flexural rigidity* (here assumed to be constant. μ is Poisson's ratio and satisfies $0<\mu<1/2$, and E is Young's modulus.

We assume that the (piecewise-smooth) boundary $\Gamma=\partial\Omega$ consists of two disjoint parts: $\Gamma=\Gamma_0\cup\Gamma_1$ with $\Gamma_1\neq\phi$ and Γ_1 relatively open in Γ. We denote by ν the unit normal to Γ pointing towards the exterior of Ω, and we set $\Sigma_0=\Gamma_0\times(0,\infty)$, $\Sigma_1=\Gamma_1\times(0,\infty)$. The boundary conditions to be considered are

$$(2.2)\qquad w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \Sigma_0,$$

$$(2.3)\qquad D[\Delta w + (1-\mu)B_1 w] = v_1 \quad \text{on } \Sigma_1,$$

$$(2.4)\qquad D[\frac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w] - \frac{\rho h^3}{12}\frac{\partial w''}{\partial \nu} = v_2 \quad \text{on } \Sigma_1,$$

where

$$B_1 w = 2\nu_1\nu_2 w_{xy} - \nu_1^2 w_{yy} - \nu_2^2 w_{xx},$$

$$B_2 w = \frac{\partial}{\partial \tau}[(\nu_1^2 - \nu_2^2)w_{xy} + \nu_1\nu_2(w_{yy} - w_{xx})]$$

and where $\tau=(-\nu_2,\nu_1)$ is a unit tangent vector. We have used (and will continue to use) the notation $w_x=\partial w/\partial x$, $w_{xy}=\partial^2 w/\partial x\partial y$, etc.

The boundary conditions (2.2) are the so-called *geometric boundary conditions* and mean that the plate is clamped along Γ_0. Conditions (2.3) and (2.4) are the *natural*, or *mechanical boundary conditions*. Condition (2.3) corresponds to bending about the axis formed by the tangent to Γ, while (2.4) incorporates both shear force in the w direction and twisting around the normal to Γ. v_1 and v_2 are the *control variables* through which the system is to be stabilized. Of course, to uniquely determine the motion, *initial conditions*

$$(2.5)\qquad w(0) = w^0, \quad w'(0) = w^1 \quad \text{in } \Omega$$

must also be specified.

The *Mindlin-Timoshenko* model arises if the Kirchhoff hypothesis is weakened by removing the assumption that the filiments of the plate remain perpendicular to the deformed middle surface, but retain the assumption that these filiments remain straight and undergo no strain in deformation. It is a system for three functions w, ψ, φ where w has the same meaning as before, $\psi \approx -(w_x+\tilde{\psi})$, $\varphi \approx -(w_y+\tilde{\varphi})$ and where $\tilde{\psi}$, $\tilde{\varphi}$ represent angles of rotation of the cross sections x=const., y=const. containing the filiment which, when the plate is in equilibrium, is perpendicular to the middle surface at the point (x,y,0). The system is the following:

$$(2.6)\qquad \frac{\rho h^3}{12}\psi'' - D[\psi_{xx} + \frac{1-\mu}{2}\psi_{yy} + \frac{1+\mu}{2}\varphi_{xy}] + K(\psi + w_x) = 0,$$

$$(2.7)\qquad \frac{\rho h^3}{12}\varphi'' - D[\varphi_{yy} + \frac{1-\mu}{2}\varphi_{xx} + \frac{1+\mu}{2}\psi_{xy}] + K(\varphi + w_y) = 0,$$

$$(2.8)\qquad \rho h w'' - K[(\psi+w_x)_x + (\varphi+w_y)_y] = 0$$

in $\Omega\times(0,\infty)$. The *geometric boundary conditions* for (2.6)-(2.8) are

$$(2.9)\qquad \psi = \varphi = w = 0 \qquad \text{on } \Sigma_0,$$

while the *mechanical boundary conditions*, applied on Σ_1, are

$$(2.10)\qquad D[\nu_1\psi_x + \mu\nu_1\varphi_y + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_2] = v_1,$$

$$(2.11)\qquad D[\nu_2\varphi_y + \mu\nu_2\psi_x + \frac{1-\mu}{2}(\psi_y + \varphi_x)\nu_1] = v_2,$$

$$(2.12)\qquad K[w_\nu + \nu_1\psi + \nu_2\varphi] = v_3$$

where $\nu=(\nu_1,\nu_2)$. Conditions (2.10) and (2.11) correspond to twisting moments while (2.12) corresponds to shear force in the w-direction. The v_i's are the control variables.

The *initial data* for (2.6)-(2.12) are

$$(2.13)\qquad \begin{aligned} &\psi(0) = \psi^0, \quad \psi'(0) = \psi^1,\\ &\varphi(0) = \varphi^0, \quad \varphi'(0) = \varphi^1,\\ &w(0) = w^0, \quad w'(0) = w^1 \qquad \text{in } \Omega. \end{aligned}$$

The constant K which appears in the above equations satisfies $K/h = kE/2(1+\mu)$. K/h is called the *coefficient of elasticity in shear*; $k>0$ is the *shear correction coefficient*. In Section 4 we shall study the convergence of the system (2.6)-(2.13) as $K\to\infty$.

Remark 2.1. Equations (2.6)-(2.8) may be formally uncoupled (see, e.g., [10, Chapter I]) to yield the following equation for the vertical displacement w:

$$(2.14)\qquad \rho h w'' - \frac{\rho h^3}{12}\Delta w'' + D\Delta^2 w + \frac{\rho h}{K}\left[\frac{h^3}{12}w'''' - D\Delta w''\right] = 0.$$

Thus (2.1) is the *formal* limit of (2.14) as $K\to\infty$.

3. Uniform Stabilization of the Mindlin-Timoshenko Model. Let ψ, φ, w be a solution to (2.6)-(2.12). The *strain energy* at time t corresponding

to this solution is given by

$$(3.1) \qquad \mathscr{G}(t) = \frac{1}{2}\int_\Omega \{D[\psi_x^2 + \varphi_y^2 + 2\mu\psi_x\varphi_y + \frac{1-\mu}{2}(\psi_y + \varphi_x)^2] + K[(\psi + w_x)^2 + (\varphi + w_y)^2]\}dxdy,$$

and the *kinetic energy* is

$$(3.2) \qquad \mathscr{K}(t) = \frac{\rho h}{2}\int_\Omega [w'^2 + \frac{h^2}{12}(\psi'^2 + \varphi'^2)]dxdy.$$

The total energy E(t) is defined to be

$$E(t) = \mathscr{G}(t) + \mathscr{K}(t).$$

We wish to determine *feedback controls* v_i in (2.10)-(2.12) such that the resulting closed-loop system is *uniformly asymptotically stable*, i.e.,

$$(3.3) \qquad E(t) \le Ce^{-\omega t}E(0)$$

for some *positive* constant ω.

In order to determine *candidates* for stabilizing feedbacks we differentiate E(t) in t. We obtain

$$E'(t) = \int_{\Gamma_1} (v_1\psi' + v_2\varphi' + v_3w')d\Gamma.$$

Therefore the feedbacks

$$(3.4) \qquad v_1 = -k_1\psi', \quad v_2 = -k_2\varphi', \quad v_3 = -k_3w'$$

with $k_i \ge 0$ introduce dissipation into the system in the sense that E(t) is nonincreasing. Actually, we can prove that if $k_i \in L^\infty(\Gamma_1)$ satisfies $k_i(x,y) \ge k_0 > 0$ a.e. on Γ_1, i=1,2,3, then under some *geometric conditions* on Γ we do, in fact, have (3.3). However, the required geometric conditions *force* $\bar\Gamma_0 \cap \bar\Gamma_1 = \phi$ if Γ is smooth and thus exclude simply connected regions Ω with smooth boundaries if $\Gamma_i \ne \phi$, i=1,2, (which we *explicitly assumed*. However, see Remark 3.3 below concerning the assumed *regularity* of solutions.)

In order to weaken the geometric restrictions and, simultaneously, to obtain *explicit* expressions for C, ω (in terms of the parameters of the

system (2.6)-(2.8) and the geometry of Γ), we alter the feedbacks (3.4) in a way which incorporates the geometric features of Γ. (This idea was first introduced in [6] in connection with stabilization of solutions of wave equations.) To this end, let us set

$$m(X) = X - X_0$$

where $X=(x,y)$, $X_0=(x_0,y_0)$, X_0 a fixed point of $\mathbb{R}^2$. We assume that X_0 may be chosen such that

(3.5) $\qquad m\cdot\nu \geq 0 \qquad$ on Γ_1,

(3.6) $\qquad m\cdot\nu \leq 0 \qquad$ on Γ_0.

We then define feedbacks by setting

(3.7) $\qquad v_1 = -(m\cdot\nu)k_1\psi', \quad v_2 = -(m\cdot\nu)k_2\varphi', \quad v_3 = -(m\cdot\nu)k_3 w'$

where $k_i \in L^\infty(\Gamma_1)$, $k_i(x,y) \geq k_0 > 0$ a.e. on Γ_1, $i=1,2,3$. We then have

THEOREM 1. *Assume that* Γ_0, Γ_1 *satisfy* (3.5), (3.6) *and that the controls* v_i *in* (2.10)-(2.11) *are given by* (3.7). *Let the initial data* (2.13) *satisfy*

$$\psi^o,\ \varphi^o,\ w^o \in H^1(\Omega), \quad \psi^o = \varphi^o = w^o = 0 \text{ on } \Gamma_0,$$
$$\psi^1,\ \varphi^1,\ w^1 \in L^2(\Omega).$$

Then there exists a constant $\omega=\omega(K)$ (*whose explicitform is given in* (3.8), (3.11) *below*) *such that*

$$\int_0^\infty E(s)ds \leq \frac{1}{\omega} E(0),$$
$$\int_t^\infty E(s)ds \leq e^{-\omega t}\int_0^\infty E(s)ds, \qquad t \geq 0.$$

COROLLARY. *Under the conditions of Theorem 1, we have*

$$E(t) \leq e^{-\omega t+1}, \qquad t \geq 1/\omega.$$

Remark. 3.1. The controls v_i of (3.7) satisfy $v_i \in L^2(\Sigma_1)$, and the problem (2.6)-(2.13), (3.7) is well-posed in the space $(H^1_{\Gamma_0}(\Omega))^3 \times (L^2(\Omega))^3$, where

$$H^1_{\Gamma_0}(\Omega) = \{\hat\varphi : \hat\varphi \in H^1(\Omega),\ \hat\varphi = 0 \text{ on } \Gamma_0\}.$$

Proof of Theorem 1. The proof is very lengthy and technical and will only be sketched here. Complete details will be published elsewhere.

For $0<\epsilon<1$, define

$$F_\epsilon(t) = Y_1(t) + (1-\epsilon)Y_2(t) + \epsilon Y_3(t)$$

where

$$Y_1(t) = \rho h\left[\frac{h^2}{12}(\psi'(t), m\cdot\nabla\psi(t)) + \frac{h^2}{12}(\varphi'(t), m\cdot\nabla\varphi(t)) + (w'(t), m\cdot\nabla w(t))\right],$$

$$Y_2(t) = \frac{\rho h^3}{12}\left[(\psi'(t),\psi(t)) + (\varphi'(t),\varphi(t))\right],$$

$$Y_3(t) = \rho h(w'(t), w(t))$$

and where

$$(\hat{\varphi},\hat{\psi}) = \int_\Omega \hat{\varphi}(x,y)\hat{\psi}(x,y)dxdy.$$

Also, for $\delta>0$, define

$$E_{\epsilon,\delta}(t) = E(t) + \delta F_\epsilon(t).$$

Since $|F_\epsilon(t)|\le CE(t)$ with C *independent of* ϵ *and* K (provided $K\ge K_0>0$), we have $E_{\epsilon,\delta}(t)\ge 0$ provided $\delta C<1$. One can prove the following (and this is *the key point*): There are positive constants c, λ_0 and λ_1 (which will be made explicit below) such that, if we choose

$$(3.8)\qquad 0 < \epsilon < \frac{1}{2(1+K\lambda_0)}, \quad 0 < \delta < \frac{2\epsilon c}{\epsilon\rho h+c\lambda_1},$$

then

$$(3.9)\qquad \frac{d}{dt}E_{\epsilon,\delta}(t) \le -\delta\epsilon E(t), \quad t\ge 0.$$

The constants c, λ_0, λ_1 are defined as follows.

$$c = \max_i \sup_{\Gamma_1} k_i(x,y).$$

In order to define λ_0 and λ_1, we introduce the notation (which will also be used in Section 4)

$$a_0(\psi,\varphi) = D\int_\Omega [\psi_x^2 + \psi_y^2 + \frac{1-\mu}{2}(\psi_y+\varphi_x)^2 + 2\mu\psi_x\varphi_y]dxdy.$$

$$a_{\Gamma_1}(\psi,\varphi,w) = \int_{\Gamma_1} m\cdot\nu\{D[\psi_x^2 + \varphi_y^2 + \frac{1-\mu}{2}(\psi_y + \varphi_x)^2 + 2\mu\psi_x\varphi_y] +$$

$$K[(\psi + w_x)^2 + (\varphi + w_y)^2]\}d\Gamma,$$

$$b(\psi,\varphi,w) = \int_{\Gamma_1} m\cdot\nu(k_1\psi^2 + k_2\varphi^2 + k_3w^2)d\Gamma.$$

Then λ_0 and λ_1 are defined by the inequalities (which incorporate the geometry of Ω)

$$\int_\Omega(\psi^2 + \varphi^2)dxdy \le \lambda_0 a_0(\psi,\varphi), \quad \forall\psi,\varphi\in H^1_{\Gamma_0}(\Omega)$$

(this last inequality is *not* true if $\Gamma_0=\phi$),

$$b(m\cdot\nabla\psi, m\cdot\nabla\varphi, m\cdot\nabla w) \le \lambda_1[a_0(\psi,\varphi) + a_{\Gamma_1}(\psi,\varphi,w)], \quad \forall\psi,\varphi,w\in H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega).$$

The conclusions of Theorem 1 are obtained from (3.9) as follows. Let $\beta>0$, multiply (3.9) by $e^{-\beta t}$ and integrate in t from t to ∞. After an integration by parts we obtain (since $E_{\epsilon,\delta}(t)\ge 0$)

$$\int_t^\infty e^{-\beta s}E(s)ds \le \frac{1}{\epsilon\delta}\, e^{-\beta t}E_{\epsilon,\delta} \le e^{-\beta t}\,\frac{1+\delta C}{\epsilon\delta}\,E(t). \tag{3.10}$$

Define

$$\omega = \frac{\epsilon\delta}{1+\delta C} \tag{3.11}$$

and let $\beta\downarrow 0$ in (3.10). The result is

$$\int_t^\infty E(s)ds \le \frac{1}{\omega}\,E(t), \qquad t \ge 0.$$

The last estimate implies the conclusions of Theorem 1.

The Corollary to Theorem 1 is a consequence of the inequality

$$\tau E(t+\tau) \le \int_t^\infty E(s)ds \le \frac{1}{\omega}\,e^{-\omega t}E(0), \quad \forall\tau>0. \tag{3.12}$$

In fact, from (3.12) we obtain

$$E(t+\tau) \le \frac{e^{\omega\tau}}{\omega\tau}\,e^{-\omega(t+\tau)}.$$

The factor $e^{\omega\tau}/\omega\tau$ has its minimum at $\tau=1/\omega$ and, for this value of τ, we obtain the estimate of the Corollary.

Remark 3.2. From (3.8) and (3.11) it follows that $\omega=\omega(K)\to 0$ as $K\to\infty$.

Remark 3.3. The proof of Theorem 1 requires that solutions be classical, i.e., $H^2(\Omega)$. However, there is a difficulty due to the possibility of singularities at points of $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, which may preclude the existence of classical solutions. Such singularities can occur regardless of the regularity of the initial data. Thus, while Theorem 1 is certainly valid if $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi$, it is proved in the opposite case *only* if one has *a priori* knowledge of the assumed regularity.

4. Limiting Behavior as $K \to \infty$. We introduce $H = L^2(\Omega)$ with the standard scalar product, and

$$V = \{\hat{\varphi}:\ \hat{\varphi} \in H^1(\Omega),\ \hat{\varphi} = 0 \text{ on } \Gamma_0\}.$$

We further define

$$a_0(\psi,\varphi;\hat{\psi},\hat{\varphi}) = D[(\psi_x,\hat{\psi}_x) + (\varphi_y,\hat{\varphi}_y) + \mu(\psi_x,\hat{\varphi}_y) + \mu(\varphi_y,\hat{\psi}_x) + \frac{1-\mu}{2}(\psi_y + \varphi_x, \hat{\psi}_y + \hat{\varphi}_x)],$$

$$a_1(\psi,\varphi,w;\hat{\psi},\hat{\varphi},\hat{w}) = (\psi + w_x, \hat{\psi} + \hat{w}_x) + (\varphi + w_y, \hat{\varphi} + \hat{w}_y),$$

$$b(\psi,\varphi,w;\hat{\psi},\hat{\varphi},\hat{w}) = \int_{\Gamma_1} m\cdot\nu(k_1\psi\hat{\psi} + k_2\varphi\hat{\varphi} + k_3 w\hat{w})d\Gamma,$$

$$c(\psi,\varphi,w;\hat{\psi},\hat{\varphi},\hat{w}) = \rho h\left[\frac{h^2}{12}(\psi,\hat{\psi}) + \frac{h^2}{12}(\varphi,\hat{\varphi}) + (w,\hat{w})\right].$$

The solution to the boundary value problem (2.6)-(2.13), (3.7) may be defined via the variational formulation

(4.1) $$\frac{d}{dt}c(\psi',\varphi',w';\hat{\psi},\hat{\varphi},\hat{w}) + a_0(\psi,\varphi;\hat{\psi},\hat{\varphi}) + Ka_1(\psi,\varphi,w;\hat{\psi},\hat{\varphi},\hat{w}) + b(\psi',\varphi',w';\hat{\psi},\hat{\varphi},\hat{w}) = 0, \quad \forall \hat{\psi},\hat{\varphi},\hat{w} \in V,$$

(4.2) $$\psi,\varphi,w \in C([0,T];V), \quad \psi',\varphi',w' \in C([0,T];H),$$

(4.3) $$[(m\cdot\nu)k_1]^{1/2}\psi'|_{\Sigma_1},\ [(m\cdot\nu)k_2]^{1/2}\varphi'|_{\Sigma_1},\ [(m\cdot\nu)k_3]^{1/2}w'|_{\Sigma_1} \in L^2(\Sigma_1),$$

(4.4) $$\psi(0) = \psi^o,\ \varphi(0) = \varphi^o,\ w(0) = w^o,$$
$$\psi'(0) = \psi^1,\ \varphi'(0) = \varphi^1,\ w'(0) = w^1.$$

If we assume that

(4.5) $$\psi^o,\varphi^o,w^o \in V, \qquad \psi^1,\varphi^1,w^1 \in H,$$

it may be shown that the problem (4.1)-(4.4) is well set. We denote its solution by ψ_K, φ_K, w_K in order to emphasize the dependence on K. We now wish to study the behavior of ψ_K, φ_K, w_K as $K\to\infty$. To do so, we adapt the technique of [10, Chapter II] to the present problem.

We choose $\hat{\psi}$, $\hat{\varphi}$, $\hat{w}$ equal to $\psi_K(t)$, $\varphi_K(t)$, $w_K(t)$, respectively, in (4.1) and integrate in t from 0 to t. (This is formal but may, for example, be justified by using a Galerkin approximation to ψ_K, φ_K, w_K.) The result is

(4.6) $$c(\psi_K'(t),\varphi_K'(t),w_K'(t)) + a_0(\psi_K(t),\varphi_K(t)) + Ka_1(\psi_K(t),\varphi_K(t),w_K(t)) + \int_0^t\int_{\Gamma_1} m\cdot\nu(k_1\psi_K'^2 + k_2\varphi_K'^2 + k_3 w_K'^2)d\Gamma d\tau$$
$$= c(\psi^1,\varphi^1,w^1) + a_0(\psi^o,\varphi^o) + Ka_1(\psi^o,\varphi^o,w^o).$$

In order to get something out of (4.6) we make the assumption (in addition to (4.5)) that

(4.7) $$a_1(\psi^o,\varphi^o,w^o) = 0,$$

i.e.,

$$\psi^o,\varphi^o\in H^1_{\Gamma_0}(\Omega), \qquad w^o\in H^2_{\Gamma_0}(\Omega),$$
$$\psi^o + w^o_x = 0, \qquad \varphi^o + w^o_y = 0,$$

where

$$H^2_{\Gamma_0}(\Omega) = \{\hat{w}: \hat{w}\in H^2(\Omega),\ \hat{w} = \hat{w}_\nu = 0 \text{ on } \Gamma_0\}$$

In addition, we assume that

$$w^1\in V.$$

Remark 4.1. A physical motivation for assumption (4.7) is as follows. The strain energy associated with a solution w of the Kirchhoff model (2.1)-(2.5) is given by

(4.8) $$\mathcal{G}(t) = \frac{1}{2}\int_\Omega D[w_{xx}^2 + w_{yy}^2 + 2\mu w_{xx}w_{yy} + 2(1-\mu)w_{xy}^2]dxdy.$$

It is seen that (4.8) is formally obtained by setting $\psi=-w_x$ and $\varphi=-w_y$ in

the expression (3.1) for the strain energy associated with the M-T model. Thus, hypothesis (4.7) may be interpreted as the requirement that *the initial strain energy be consistent with the Kirchhoff model.*

Assuming (4.7), we obtain from (4.6) that

(4.9) ψ_K, φ_K are bounded in $L^\infty(0,T;V)$,

ψ_K', φ_K', w_K' are bounded in $L^\infty(0,T;H)$,

$[(m\cdot\nu)k_1]^{1/2}\psi_K'$, $[(m\cdot\nu)k_2]^{1/2}\varphi_K'$, $[(m\cdot\nu)k_3]^{1/2}w_K'$

are bounded in $L^2(0,T;L^2(\Gamma_1))$,

(4.10) $\sqrt{K}(\psi_K+w_{Kx})$, $\sqrt{K}(\varphi_K+w_{Ky})$

are bounded in $L^\infty(0,T;H)$.

From (4.9), (4.10) follows that we also have

w_{Kx}, w_{Ky} are bounded in $L^\infty(0,T;H)$.

Therefore, we may choose a subsequence, still denoted by ψ_K, φ_K, w_K, such that

(4.11) ψ_K, φ_K, $w_K \to \psi$, φ, w weak* in $L^\infty(0,T;V)$,

(4.12) ψ_K', φ_K', $w_K' \to \psi'$, φ', w' weak* in $L^\infty(0,T;H)$,

(4.13) (ψ_K+w_{Kx}), $(\varphi_K+w_{Ky}) \to 0$ strongly in $L^\infty(0,T;H)$,

(4.14) $[(m\cdot\nu)k_1]^{1/2}\psi_K' \to [(m\cdot\nu)k_1]^{1/2}\psi'$ weakly in $L^2(0,T;L^2(\Gamma_1))$,

(4.15) $[(m\cdot\nu)k_2]^{1/2}\varphi_K' \to [(m\cdot\nu)k_2]^{1/2}\varphi'$ weakly in $L^2(0,T;L^2(\Gamma_1))$,

(4.16) $[(m\cdot\nu)k_3]^{1/2}w_K' \to [(m\cdot\nu)k_3]^{1/2}w'$ weakly in $L^2(0,T;L^2(\Gamma_1))$.

From (4.11),(4.13) it follows that

(4.17) $\psi + w_x = 0$, $\qquad \varphi + w_y = 0$,

(4.18) $\psi' + w_x' = 0$, $\qquad \varphi' + w_y' = 0$.

Therefore

(4.19) $w\in L^\infty(0,T;H^2_{\Gamma_0}(\Omega))$, $\quad w'\in L^\infty(0,T;H^1_{\Gamma_0}(\Omega))$.

We also obtain from the limiting process that

$$w(0) = w^0, \qquad w'(0)= w^1.$$

Next, in (4.1) we choose $\hat{w}\in H^2_{\Gamma_0}(\Omega)$ and $\hat{\psi}$, $\hat{\varphi}$ according to

$$\hat{\psi} = -\hat{w}_x, \qquad \hat{\varphi} = -\hat{w}_y.$$

We obtain the equation

$$(4.20) \qquad \frac{d}{dt} c(\psi_K', \varphi_K', w_K'; -\hat{w}_x, -\hat{w}_y, \hat{w}) + a_0(\psi_K, \varphi_K; -\hat{w}_x, -\hat{w}_y) + b(\psi_K', \varphi_K', w_K'; -\hat{w}_x, -\hat{w}_y, \hat{w}) = 0, \quad \forall \hat{w}\in H^2_{\Gamma_0}(\Omega).$$

Using (4.11)-(4.16) we may pass to the limit (in the sense of distributions) in (4.20). Let us write out the limit equation, using (4.17), (4.18). We obtain

$$(4.21) \qquad \frac{d}{dt}[\rho h(w', \hat{w}) + \frac{\rho h^3}{12}(\nabla w', \nabla\hat{w})] + D\int_\Omega [w_{xx}\hat{w}_{xx} + w_{yy}\hat{w}_{yy} + \mu(w_{xx}\hat{w}_{yy} + w_{yy}\hat{w}_{xx}) + 2(1-\mu)w_{xy}\hat{w}_{xy}]dxdy + \int_{\Gamma_1} m\cdot\nu(k_1 w_x'\hat{w}_x + k_2 w_y'\hat{w}_y + k_3 w'\hat{w})d\Gamma = 0, \qquad \forall \hat{w}\in H^2_{\Gamma_0}(\Omega).$$

$$(4.22) \qquad w(0) = w^0\in H^2_{\Gamma_0}(\Omega), \quad w'(0) = w^1\in H^1_{\Gamma_0}(\Omega).$$

But (4.21), (4.22), (4.19) has a unique solution and are exactly the variational formulation of the boundary value problem

$$(4.23) \qquad \rho h w'' - \frac{\rho h^3}{12}\Delta w'' + D\Delta^2 w = 0 \quad \text{in } \Omega\times(0,T),$$

$$(4.24) \qquad w = \frac{\partial w}{\partial\nu} = 0 \qquad \text{on } \Sigma_0,$$

$$(4.25) \qquad D[\Delta w + (1-\mu)B_1 w] = -(m\cdot\nu)(k_1\nu_1 w_x' + k_2\nu_2 w_y') \quad \text{on } \Sigma_1,$$

$$(4.26) \qquad D[\frac{\partial\Delta w}{\partial\nu} + (1-\mu)B_2 w] - \frac{\rho h^3}{12}\frac{\partial w''}{\partial\nu} = (m\cdot\nu)k_3 w' - \frac{\partial}{\partial\tau}[m\cdot\nu(-k_1\nu_2 w_x' + k_2\nu_1 w_y')] \quad \text{on } \Sigma_1,$$

$$(4.27) \qquad w(0) = w^0, \quad w'(0) = w^1 \qquad \text{in } \Omega.$$

Remark 4.2. It may be shown by methods similar to those of Section 3 that the problem (4.23)-(4.27) is *uniformly asymptotically stable*. Thus this problem has a positive energy decay rate ω^* which is *not* the limit of $\omega(K)$ as $K\to\infty$ (defined in (3.8), (3.11)).

Remark 4.3. If $k_1=k_2=k^*$, the right sides of (4.25), (4.26) reduce to

(4.28) $\qquad -(m\cdot\nu)k^{*}\frac{\partial w'}{\partial\nu}$

and

(4.29) $\qquad (m\cdot\nu)k_3 w' - \frac{\partial}{\partial\tau}[(m\cdot\nu)k^{*}\frac{\partial w'}{\partial\tau}]$

respectively. When the inertia term $-(\rho h^3/12)\Delta w''$ in (4.23) (and the corresponding term in (4.26)) are absent, uniform asymptotic stability was proved in [9] using the feedbacks defined in (4.28), (4.29) with $k^{*}=0$.

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